

# MARCINKIEWICZ THEOREM FOR LORENTZ GAMMA SPACES

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**ABSTRACT.** This paper is devoted to the interpolation principle between spaces of weak type. We characterize interpolation spaces between two Marcinkiewicz spaces in terms of Hardy type operators involving suprema. We study general properties of such operators and their behavior on Lorentz gamma spaces.

## 1. INTRODUCTION

Let  $\mathcal{R} = (\mathcal{R}, \mu)$  be non-atomic  $\sigma$ -finite measure space with  $\mu(\mathcal{R}) = R$ , where  $0 < R \leq \infty$ . Let  $\mathcal{M}(\mathcal{R}, \mu)$  denote the collection of all extended real-valued  $\mu$ -measurable and a.e. finite functions on  $\mathcal{R}$ .

This paper deals with Marcinkiewicz interpolation theorem between spaces of weak type where the norm is defined by

$$\|f\|_{M_\varphi(\mathcal{R})} = \sup_{0 < s < R} \varphi(s) f^{**}(s).$$

Here  $\varphi$  is so-called quasiconcave function (for the definition see Section 2), the double stars stand for the maximal function defined as a Hardy average of  $f^*$ ,

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds,$$

in which  $f^*$  represents the non-increasing rearrangement of  $f$ , given by

$$f^*(t) = \inf\{\lambda > 0; \mu(\{x \in \mathcal{R}; |f(x)| > \lambda\}) \leq t\}, \quad t \in [0, R).$$

The collection of all functions  $f \in \mathcal{M}(\mathcal{R}, \mu)$  with  $\|f\|_{M_\varphi(\mathcal{R})}$  finite is called Marcinkiewicz space  $M_\varphi(\mathcal{R}, \mu)$ .

In our main result we prove that the boundedness of a certain operator is ensured by that of the supremum operators or, more precisely, Hardy-type operators involving suprema  $S_\varphi$  and  $T_\psi$  defined by

$$S_\varphi f(t) = \frac{1}{\varphi(t)} \sup_{0 < s < t} \varphi(s) f^*(s), \quad t \in (0, R),$$

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$$T_\psi f(t) = \frac{1}{\psi(t)} \sup_{t < s < R} \psi(s) f^*(s), \quad t \in (0, R),$$

where  $\varphi$  and  $\psi$  are quasiconcave functions. Such a result was first proved by Dmitriev and Kreĭn in [3]; however, the supremum operators appeared only implicitly. Later, Kerman and Pick in [6] and [7] showed the equivalence of the boundedness of the operators of such kind and certain Sobolev-type embeddings and they also used their result in the search of optimal pairs of r.i. spaces for which these embeddings hold. Consequently, Kerman, Phipps and Pick in [5] found simple criteria for the boundedness of the supremum operators on Orlicz spaces and Lorentz Gamma spaces and they obtained corresponding Marcinkiewicz interpolation theorems. However, all the above-mentioned results concern only power functions in place of  $\varphi$ . In this work, we want to fill this gap.

The principal innovation of this paper consists not only in a significant extension of the known results but also in the new and more elegant comprehensive approach that enables us to establish proofs which are more enlightening and illustrative and less technical than those applied in earlier works.

We will work in the general setting of rearrangement-invariant (r.i. for short) Banach function spaces  $X(\mathcal{R}, \mu)$  as collections of all  $\mu$ -measurable functions finite a.e. on  $\mathcal{R}$  such that  $\|f\|_{X(\mathcal{R}, \mu)}$  is finite.

One can define an r.i. space  $X(\mathcal{R}, \mu)$  on a general measure space  $(\mathcal{R}, \mu)$  using rearrangement invariance of the given r.i. space  $X(0, R)$ ,

$$\|f\|_{X(\mathcal{R}, \mu)} = \|f^*\|_{X(0, R)}, \quad f \in \mathcal{M}(\mathcal{R}, \mu).$$

On the other hand, there is also a representation of each norm of a given r.i. space  $X(\mathcal{R}, \mu)$  by r.i. norm on interval due to the Luxemburg representation theorem. For further information regarding r.i. norms see [1, Chapter 1 and 2]. At the places where no confusion is likely to happen, we shall use a shorter form  $X(\mathcal{R})$  instead of  $X(\mathcal{R}, \mu)$  and  $X$  in the case when  $(\mathcal{R}, \mu)$  is the interval  $[0, R)$  equipped with Lebesgue measure.

We also exhibit the general properties of the supremum operators  $S_\varphi$  and  $T_\psi$  like the endpoint embeddings in the r.i. class (Section 3) or the relation to the maximal function (Section 4). It turns out that a certain averaging condition on the quasiconcave function plays a key role here. It reads as

$$\frac{1}{t} \int_0^t \frac{ds}{\varphi(s)} \lesssim \frac{1}{\varphi(t)}, \quad t \in (0, R).$$

We shall refer to this relation as a  $B$ -condition and write  $\varphi \in B$ . More details about quasiconcave functions and  $B$ -condition can be found in Section 2.

Our principal result now reads as follows.

**Theorem 1.1.** *Let  $\mathcal{R}_1 = (\mathcal{R}_1, \mu_1)$  and  $\mathcal{R}_2 = (\mathcal{R}_2, \mu_2)$  be non-atomic  $\sigma$ -finite measure spaces for which  $\mu_1(\mathcal{R}_1) = \mu_2(\mathcal{R}_2) = R$ . Suppose that a quasilinear operator  $T$  satisfies*

$$T: M_\varphi(\mathcal{R}_1) \rightarrow M_\varphi(\mathcal{R}_2) \quad \text{and} \quad T: M_\psi(\mathcal{R}_1) \rightarrow M_\psi(\mathcal{R}_2)$$

for quasiconcave functions  $\varphi, \psi$  defined on  $[0, R)$ , both satisfying the B-condition and let  $X_i(\mathcal{R}_i)$ ,  $i = 1, 2$ , be r.i. spaces satisfying

$$M_\varphi(\mathcal{R}_i) \cap M_\psi(\mathcal{R}_i) \subset X_i(\mathcal{R}_i) \subset M_\varphi(\mathcal{R}_i) + M_\psi(\mathcal{R}_i), \quad i = 1, 2.$$

Then

$$T: X_1(\mathcal{R}_1) \rightarrow X_2(\mathcal{R}_2)$$

whenever

$$S_\varphi: X_1(0, R) \rightarrow X_2(0, R) \quad \text{and} \quad T_\psi: X_1(0, R) \rightarrow X_2(0, R). \quad (1.1)$$

Our next result concerns the criteria to guarantee (1.1) in specific class of r.i. spaces, namely in the classical Lorentz gamma spaces  $\Gamma_\phi^p(\mathcal{R})$  where the norm is given as

$$\|f\|_{\Gamma_\phi^p(\mathcal{R})} = \left( \int_0^R [f^{**}(t)]^p \phi(t) dt \right)^{\frac{1}{p}}.$$

Here  $1 \leq p < \infty$  and  $\phi$  is some positive and locally integrable function, so-called weight. We require  $\int_1^\infty s^{-p} \phi(s) ds < \infty$  when  $R = \infty$  and  $\int_0^R s^{-p} \phi(s) ds = \infty$  when  $R < \infty$  otherwise  $\Gamma_\phi^p = \{0\}$  in the first case and  $\Gamma_\phi^p = L^1$  in the second one. Such requirements are called nontriviality conditions.

If we deal with the operator  $S_\varphi$  acting between Lorentz gamma spaces with discontinuous  $\varphi$ , we moreover admit additional nontriviality conditions, i.e, we assume

$$\int_0^R \varphi^{-p}(s) \phi(s) ds < \infty \quad (1.2)$$

and

$$\lim_{t \rightarrow 0^+} t^p \int_t^R s^{-p} \phi(s) ds > 0. \quad (1.3)$$

As we explain in Remark 5.1, such requirements are necessary and cause no loss of generality.

**Theorem 1.2.** *Let  $\mathcal{R}_1 = (\mathcal{R}_1, \mu_1)$  and  $\mathcal{R}_2 = (\mathcal{R}_2, \mu_2)$  be non-atomic  $\sigma$ -finite measure spaces with  $\mu_1(\mathcal{R}_1) = \mu_2(\mathcal{R}_2) = R$ ,  $\varphi$  and  $\psi$  be quasiconcave functions defined on  $[0, R)$  satisfying the B-condition,  $\phi_1$  and  $\phi_2$  be nontrivial weights on  $(0, R)$ . In the case  $\varphi$  is not continuous, let, in addition,  $\phi_1$  and  $\phi_2$  satisfy (1.3) and (1.2) respectively. Let  $p$  be an index,  $1 \leq p < \infty$ , such that*

$$M_\varphi(\mathcal{R}_i) \cap M_\psi(\mathcal{R}_i) \subset \Gamma_{\phi_i}^p(\mathcal{R}_i) \subset M_\varphi(\mathcal{R}_i) + M_\psi(\mathcal{R}_i), \quad i = 1, 2.$$

Suppose  $T$  is a quasilinear operator that satisfies

$$T: M_\varphi(\mathcal{R}_1) \rightarrow M_\varphi(\mathcal{R}_2) \quad \text{and} \quad T: M_\psi(\mathcal{R}_1) \rightarrow M_\psi(\mathcal{R}_2);$$

then, a sufficient condition for the embedding

$$T: \Gamma_{\phi_1}^p(\mathcal{R}_1) \rightarrow \Gamma_{\phi_2}^p(\mathcal{R}_2)$$

is as follows

$$\sup_{0 < t < R} \frac{\psi^p(t) \int_0^t \psi^{-p}(s) \phi_2(s) \, ds + \varphi^p(t) \int_t^R \varphi^{-p}(s) \phi_2(s) \, ds}{\int_0^t \phi_1(s) \, ds + t^p \int_t^R s^{-p} \phi_1(s) \, ds} < \infty. \quad (1.4)$$

The proof of this result follows from a characterization of the boundedness of the supremum operators  $S_\varphi$  and  $T_\varphi$  between two Lorentz gamma spaces, of independent interest, formulated in the following two theorems.

**Theorem 1.3.** *Let  $1 \leq p < \infty$ , let  $\varphi$  be a quasiconcave function on  $[0, R)$  satisfying the  $B$ -condition and let  $\phi_1, \phi_2$  be nontrivial weights on  $(0, R)$ . In the case  $\varphi$  is not continuous, let, in addition,  $\phi_1$  and  $\phi_2$  satisfy (1.3) and (1.2) respectively. Then*

$$S_\varphi : \Gamma_{\phi_1}^p(0, R) \rightarrow \Gamma_{\phi_2}^p(0, R) \quad (1.5)$$

holds if and only if

$$\sup_{0 < t < R} \frac{\int_0^t \phi_2(s) \, ds + \varphi^p(t) \int_t^R \varphi^{-p}(s) \phi_2(s) \, ds}{\int_0^t \phi_1(s) \, ds + t^p \int_t^R s^{-p} \phi_1(s) \, ds} < \infty. \quad (1.6)$$

**Theorem 1.4.** *Let  $1 \leq p < \infty$ , let  $\psi$  be a quasiconcave function on  $[0, R)$  satisfying the  $B$ -condition and let  $\phi_1, \phi_2$  be nontrivial weights on  $(0, R)$ . Then*

$$T_\psi : \Gamma_{\phi_1}^p(0, R) \rightarrow \Gamma_{\phi_2}^p(0, R) \quad (1.7)$$

holds if and only if

$$\sup_{0 < t < R} \frac{\psi^p(t) \int_0^t \psi^{-p}(s) \phi_2(s) \, ds + t^p \int_t^R s^{-p} \phi_2(s) \, ds}{\int_0^t \phi_1(s) \, ds + t^p \int_t^R s^{-p} \phi_1(s) \, ds} < \infty. \quad (1.8)$$

The proofs of these results appear in Section 5. One may also notice that we sometimes avoid stating the results in the full generality. For instance, one may try to extend Theorems 1.3 and 1.4 to various exponents on the left and the right hand sides or avoid the  $B$ -condition. The reason is similar here; the general situation can be treated by discretization methods while we want to keep the approach as simple as possible.

## 2. QUASICONCAVE FUNCTIONS

Let us recall that if a non-negative function defined on  $[0, R)$ ,  $\varphi$ , satisfies

- (i)  $\varphi(t) = 0$  if and only if  $t = 0$ ;
- (ii)  $\varphi$  is non-decreasing;
- (iii)  $\varphi(t)/t$  is non-increasing on  $(0, R)$ ,

then  $\varphi$  is said to be *quasiconcave*. If we denote  $\tilde{\varphi}(t) = t/\varphi(t)$  for  $t \in (0, R)$  and  $\tilde{\varphi}(0) = 0$  then  $\tilde{\varphi}$  is also a quasiconcave function. We say that  $\tilde{\varphi}$  is complementary function to  $\varphi$ .

A quasiconcave function  $\varphi$  is continuous in every positive argument from its domain. Any jump at such a point would lead to the contradiction with the monotonicity of

complementary function  $\tilde{\varphi}$  or  $\varphi$  itself. Only possible point of discontinuity of quasiconcave functions is zero.

In the next theorem we will give a equivalent form of the  $B$ -condition. The idea is based on [9, Lemma 2.3]. Here and in the sequel we will use the notation  $A \lesssim B$  if  $A \leq CB$  where  $C$  is a constant independent of all quantities obtained in  $A$  and  $B$ . In the case  $A \lesssim B$  and  $B \lesssim A$  we will use  $A \simeq B$ .

**Theorem 2.1** (Characterization of  $B$ -condition). *Let  $\varphi$  be a quasiconcave function on  $[0, R)$ . Then the following conditions are equivalent.*

- (i)  $\varphi$  satisfies  $B$ -condition;
- (ii) It holds

$$\int_0^t \tilde{\varphi}(s) \frac{ds}{s} \lesssim \tilde{\varphi}(t), \quad t \in (0, R);$$

- (iii) There exists a constant  $c \in (0, 1)$  such that

$$\inf_{0 < t < R} \frac{\tilde{\varphi}(t)}{\tilde{\varphi}(ct)} > 1.$$

**Proof.** (i) is equivalent to (ii) simply by the definition of  $\tilde{\varphi}$ . Next, suppose that (iii) holds. There exists a constant  $r > 1$  such that

$$\tilde{\varphi}(t) \geq r\tilde{\varphi}(ct), \quad t \in (0, R).$$

Using this inequality iteratively for  $t, ct, c^2t, \dots$ , we get

$$\tilde{\varphi}(t) \geq r\tilde{\varphi}(ct) \geq r^k \tilde{\varphi}(c^k t), \quad t \in (0, R), \quad k \in \mathbb{N}.$$

Now, we slice the integration domain of the integral in the first condition and since  $\tilde{\varphi}$  is increasing, we get

$$\begin{aligned} \int_0^t \tilde{\varphi}(s) \frac{ds}{s} &= \sum_{k=0}^{\infty} \int_{c^{k+1}t}^{c^k t} \tilde{\varphi}(s) \frac{ds}{s} \leq \sum_{k=0}^{\infty} \tilde{\varphi}(c^k t) \int_{c^{k+1}t}^{c^k t} \frac{ds}{s} \\ &= \log(1/c) \sum_{k=0}^{\infty} \tilde{\varphi}(c^k t) \leq \tilde{\varphi}(t) \log(1/c) \sum_{k=0}^{\infty} r^{-k} \\ &= \frac{r \log(1/c)}{r-1} \tilde{\varphi}(t), \quad t \in (0, R), \end{aligned}$$

so (ii) holds.

In the opposite direction let us assume that (iii) is not satisfied and (ii) is; in other words, for some positive constant  $K$ ,

$$\int_0^t \tilde{\varphi}(s) \frac{ds}{s} \leq K\tilde{\varphi}(t), \quad t \in (0, R).$$

Now, fix an arbitrary  $r > 1$ . Then, for every constant  $c \in (0, 1)$  there exists  $t \in (0, R)$  such that

$$\tilde{\varphi}(t) < r\tilde{\varphi}(ct).$$

Using all this, we obtain

$$\begin{aligned} K\tilde{\varphi}(t) &\geq \int_0^t \tilde{\varphi}(s) \frac{ds}{s} \geq \int_{ct}^t \tilde{\varphi}(s) \frac{ds}{s} \\ &\geq \tilde{\varphi}(ct) \log(1/c) > \frac{1}{r} \tilde{\varphi}(t) \log(1/c). \end{aligned}$$

Thus, we have a contradiction since  $Kr \geq \log(1/c)$  cannot hold for every  $c \in (0, 1)$  and the proof is complete.  $\square$

**Lemma 2.2.** *Let  $\varphi$  be a quasiconcave function on  $[0, R)$ . Then*

$$\sup_{0 < t < R} \varphi(t) f^{**}(t) \simeq \sup_{0 < t < R} \varphi(t) f^*(t) \quad (2.1)$$

for every measurable  $f$  if and only if  $\varphi \in B$ .

**Proof.** Necessity follows immediately by setting  $f = f^* = 1/\varphi$ .

Now suppose that  $\varphi \in B$ . Since  $f^* \leq f^{**}$  the left hand side of (2.1) dominates the right hand side of (2.1). For the opposite inequality denote the right hand side of (2.1) by  $M$ . We then have

$$f^*(t) \leq M \frac{1}{\varphi(t)}, \quad t \in (0, R).$$

Integrating this inequality over  $(0, s)$  and dividing by  $s$  we get

$$f^{**}(s) \leq \frac{M}{s} \int_0^s \frac{dt}{\varphi(t)} \lesssim M \frac{1}{\varphi(s)}, \quad s \in (0, R),$$

hence

$$\sup_{0 < s < R} \varphi(s) f^{**}(s) \lesssim M$$

as we wished to show.  $\square$

**Remark 2.3.** *Note that for a given measurable function  $f$ , both  $T_\psi f$  and  $S_\varphi f$  are non-increasing functions. Indeed,*

$$\begin{aligned} S_\varphi f(t) &= \frac{1}{\varphi(t)} \sup_{0 < s < t} \varphi(s) \sup_{s < y < R} f^*(y) \\ &= \frac{1}{\varphi(t)} \sup_{0 < y < R} f^*(y) \sup_{0 < s < \min\{t, y\}} \varphi(s) \\ &= \sup_{0 < y < R} f^*(y) \min \left\{ 1, \frac{\varphi(y)}{\varphi(t)} \right\} \end{aligned}$$

which is clearly non-increasing. The case concerning  $T_\psi f$  is obvious.

## 3. ENDPOINT ESTIMATES

**Lemma 3.1.** *Let  $\psi$  be a quasiconcave function on  $[0, R)$ . Then*

(i)

$$T_\psi: L^1 \rightarrow L^1 \quad \text{if and only if} \quad \psi \in B;$$

(ii)

$$T_\psi: M_\psi \rightarrow M_\psi \quad \text{if and only if} \quad \psi \in B.$$

**Proof.** For the necessity of the  $B$ -condition we just put  $f = \chi_{(0,t)}$ . The calculations are straightforward. For the sufficiency in (i) we split the integration in two parts, namely

$$\begin{aligned} \|T_\psi f\|_{L^1} &= \int_0^R \frac{1}{\psi(t)} \sup_{t < s < R} \psi(s) f^*(s) \, dt \\ &\leq \int_0^R \frac{1}{\psi(t)} \sup_{t < s < R} (\psi(s) - \psi(t)) f^*(s) \, dt + \int_0^R \sup_{t < s < R} f^*(s) \, dt \\ &= \text{I} + \text{II}. \end{aligned}$$

The second part equals to the  $L^1$  norm of  $f$ , while the first part needs some estimates. We have

$$\begin{aligned} \text{I} &= \int_0^R \frac{1}{\psi(t)} \sup_{t < s < R} \left( \int_t^s \psi'(y) \, dy \right) f^*(s) \, dt \\ &\leq \int_0^R \frac{1}{\psi(t)} \sup_{t < s < R} \left( \int_t^s \psi'(y) f^*(y) \, dy \right) dt \\ &= \int_0^R \frac{1}{\psi(t)} \int_t^R \psi'(y) f^*(y) \, dy \, dt \\ &= \int_0^R \psi'(y) f^*(y) \int_0^y \frac{dt}{\psi(t)} \, dy && \text{(by the Fubini theorem)} \\ &\lesssim \int_0^R \frac{y}{\psi(y)} \psi'(y) f^*(y) \, dy && \text{(since } \psi \in B) \\ &\leq \int_0^R f^*(y) \, dy. && \text{(by quasiconcavity)} \end{aligned}$$

This completes the proof of the part (i). For the sufficiency in the part (ii), recall that  $T_\psi f$  is non-increasing and hence we have

$$\begin{aligned}
\|T_\psi f\|_{M_\psi} &= \sup_{0 < t < R} \psi(t) (T_\psi f)^{**}(t) \\
&= \sup_{0 < t < R} \frac{\psi(t)}{t} \int_0^t \frac{1}{\psi(s)} \sup_{s < y < R} \psi(y) f^*(y) \, ds \\
&\leq \sup_{0 < t < R} \frac{\psi(t)}{t} \int_0^t \frac{1}{\psi(s)} \sup_{0 < y < R} \psi(y) f^{**}(y) \, ds \\
&= \|f\|_{M_\psi} \sup_{0 < t < R} \frac{\psi(t)}{t} \int_0^t \frac{ds}{\psi(s)}
\end{aligned}$$

and the last supremum is finite because of the  $B$ -condition for  $\psi$ . □

**Lemma 3.2.** *Let  $\varphi$  be a quasiconcave function on  $[0, R)$ . Then*

(i)

$$S_\varphi: M_\varphi \rightarrow M_\varphi \quad \text{if and only if} \quad \varphi \in B;$$

(ii)

$$S_\varphi: L^\infty \rightarrow L^\infty \quad \text{for every quasiconcave } \varphi.$$

**Proof.** Let us consider part (i). For the necessity we set  $f = \chi_{(0,a)}$ . We obtain

$$S_\varphi \chi_{(0,a)}(t) = \min \left\{ 1, \frac{\varphi(a)}{\varphi(t)} \right\}, \quad t \in (0, R), \quad a \in (0, R),$$

and thus for every  $a \in (0, R)$  we have

$$\begin{aligned}
\|S_\varphi \chi_{(0,a)}\|_{M_\varphi} &= \sup_{0 < t < R} \frac{\varphi(t)}{t} \int_0^t \min \left\{ 1, \frac{\varphi(a)}{\varphi(s)} \right\} \, ds \\
&= \sup_{0 < t < R} \varphi(t) \chi_{(0,a)}(t) + \frac{\varphi(t)}{t} \left( a + \varphi(a) \int_a^t \frac{ds}{\varphi(s)} \right) \chi_{(a,R)}(t) \\
&\geq \varphi(a) \sup_{a < t < R} \frac{\varphi(t)}{t} \int_a^t \frac{ds}{\varphi(s)}.
\end{aligned}$$

Clearly  $\|\chi_{(0,a)}\|_{M_\varphi} = \varphi(a)$  and since  $S_\varphi$  is bounded on  $M_\varphi$  we get

$$\varphi(a) \sup_{a < t < R} \frac{\varphi(t)}{t} \int_a^t \frac{ds}{\varphi(s)} \leq \varphi(a), \quad a \in (0, R).$$

The term  $\varphi(a)$  cancels and by taking the limit  $a \rightarrow 0^+$  we get the  $B$ -condition.



Now suppose that  $\varphi \in B$ . Taking Lemma 2.2 and the monotonicity of  $S_\varphi f$  into account, we have

$$\begin{aligned} \|S_\varphi f\|_{M_\varphi} &= \sup_{0 < t < R} \varphi(t) (S_\varphi f)^{**}(t) \simeq \sup_{0 < t < R} \varphi(t) (S_\varphi f)^*(t) \\ &= \sup_{0 < t < R} \varphi(t) (S_\varphi f)(t) = \sup_{0 < t < R} \sup_{0 < s < t} \varphi(s) f^*(s) \\ &\simeq \sup_{0 < s < R} \varphi(s) f^{**}(s) = \|f\|_{M_\varphi}. \end{aligned}$$

Part (ii) is trivial. □

#### 4. STARFALLS

**Lemma 4.1.** *Let  $\psi$  be a quasiconcave function on  $[0, R)$ . Then*

$$(i) \quad (T_\psi f)^{**} \lesssim T_\psi f + f^{**} \quad \text{if and only if} \quad \psi \in B;$$

$$(ii) \quad T_\psi f^{**} \simeq T_\psi f + f^{**} \quad \text{if and only if} \quad \psi \in B.$$

**Proof.** To prove the necessity, we test the inequalities by characteristic function  $f = \chi_{(0,a)}$ . We compute

$$T_\psi \chi_{(0,a)}(t) = \chi_{(0,a)}(t) \frac{\psi(a)}{\psi(t)}$$

and

$$T_\psi \chi_{(0,a)}^{**}(t) = \chi_{(0,a)}(t) \frac{\psi(a)}{\psi(t)} + \chi_{(a,R)}(t) \frac{a}{t}$$

and also

$$(T_\psi \chi_{(0,a)})^{**}(t) = \chi_{(0,a)}(t) \frac{\psi(a)}{t} \int_0^t \frac{ds}{\psi(s)} + \chi_{(a,R)}(t) \frac{\psi(a)}{t} \int_0^a \frac{ds}{\psi(s)}$$

for every pair  $a$  and  $t$  in  $(0, R)$ . The necessity of the  $B$ -condition then follows by comparing appropriate quantities for arbitrary  $t < a$ .

To prove the sufficiency in (i), we divide the outer integral into three parts.

$$\begin{aligned} \frac{1}{t} \int_0^t \frac{1}{\psi(y)} \sup_{y < s < R} \psi(s) f^*(s) dy &\leq \frac{1}{t} \int_0^t \frac{1}{\psi(y)} \sup_{y < s < t} (\psi(s) - \psi(y)) f^*(s) dy \\ &\quad + \frac{1}{t} \int_0^t \sup_{y < s < t} f^*(s) dy \\ &\quad + \frac{1}{t} \int_0^t \frac{1}{\psi(y)} \sup_{t < s < R} \psi(s) f^*(s) dy \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

The first term can be treated in the same way as in the proof of Lemma 3.1, part (i). We get  $\text{I} \lesssim f^{**}(t)$ . The term II clearly equals  $f^{**}(t)$ . Finally, since  $\psi \in B$ ,

$$\text{III} \lesssim \frac{1}{\psi(t)} \sup_{t < s < R} \psi(s) f^*(s) = T_\psi f(t), \quad t \in (0, R).$$

Adding all these estimates together we have

$$(T_\psi f)^{**}(t) \lesssim f^{**}(t) + T_\psi f(t), \quad t \in (0, R).$$

Let us show the equivalence (ii) assuming  $\psi \in B$ . One inequality is obvious since  $f^{**} \leq T_\psi f^{**}$ . The reversed inequality can be observed by the splitting argument similar to that in part (i). For  $t \in (0, R)$ , we have

$$\begin{aligned} T_\psi f^{**}(t) &= \frac{1}{\psi(t)} \sup_{t < s < R} \frac{\psi(s)}{s} \int_0^s f^*(y) \, dy \\ &\leq \frac{1}{\psi(t)} \sup_{t < s < R} \frac{\psi(s)}{s} \int_t^s f^*(y) \, dy + \frac{1}{\psi(t)} \sup_{t < s < R} \frac{\psi(s)}{s} \int_0^t f^*(y) \, dy \\ &= \text{I} + \text{II}. \end{aligned}$$

Now, we can continue by

$$\begin{aligned} \text{I} &= \frac{1}{\psi(t)} \sup_{t < s < R} \frac{\psi(s)}{s} \int_t^s \psi(y) f^*(y) \frac{dy}{\psi(y)} \\ &\leq \frac{1}{\psi(t)} \sup_{t < y < R} \psi(y) f^*(y) \sup_{t < s < R} \frac{\psi(s)}{s} \int_t^s \frac{dy}{\psi(y)} \quad (\text{by taking the supremum out}) \\ &\leq T_\psi f(t) \sup_{0 < s < R} \frac{\psi(s)}{s} \int_0^s \frac{dy}{\psi(y)} \quad (\text{by taking } t = 0) \\ &\lesssim T_\psi f(t) \quad (\text{since } \psi \in B) \end{aligned}$$

and surely  $\text{II} = f^{**}(t)$ .  $\square$

**Lemma 4.2.** *Let  $\varphi$  be a quasiconcave function on  $[0, R)$ . Then*

(i)

$$(S_\varphi f)^{**} \lesssim S_\varphi f^{**} \quad \text{if and only if } \varphi \in B;$$

(ii)

$$S_\varphi f^{**} \lesssim S_\varphi f \quad \text{if and only if } \varphi \in B.$$

**Proof.** Part (i). The necessity follows by plugging  $f = \chi_{(0,a)}$  into the inequality. We have

$$S_\varphi \chi_{(0,a)}(t) = S_\varphi \chi_{(0,a)}^{**}(t) = \min \left\{ 1, \frac{\varphi(a)}{\varphi(t)} \right\}, \quad t \in (0, R), \quad a \in (0, R).$$

Similarly as in the proof of Lemma 3.2 we calculate

$$(S_\varphi \chi_{(0,a)})^{**}(t) = \chi_{(0,a)}(t) + \frac{1}{t} \left( a + \varphi(a) \int_a^t \frac{ds}{\varphi(s)} \right) \chi_{(a,R)}(t), \quad a \in (0, R), \quad t \in (0, R),$$

hence for  $t > a$  we have

$$(S_\varphi \chi_{(0,a)})^{**}(t) \geq \frac{\varphi(a)}{t} \int_a^t \frac{ds}{\varphi(s)},$$

therefore for those  $t$  and  $a$  we have

$$\frac{\varphi(a)}{t} \int_a^t \frac{ds}{\varphi(s)} \lesssim \frac{\varphi(a)}{\varphi(t)}.$$

The term  $\varphi(a)$  cancels and by taking the limit  $a \rightarrow 0^+$  we obtain the  $B$ -condition.

On the other side, we have

$$\begin{aligned} \varphi(t)(S_\varphi f)^{**}(t) &\leq \sup_{0 < s < t} \varphi(s)(S_\varphi f)^{**}(s) \\ &= \|S_\varphi f\|_{M_\varphi(0,t)} \lesssim \|f\|_{M_\varphi(0,t)} = \sup_{0 < s < t} \varphi(s)f^{**}(s) \end{aligned}$$

thanks to Lemma 3.2. Dividing by  $\varphi(t)$  we get the result.

Part (ii) follows immediately with the help of Lemma 2.2 by

$$\varphi(t)S_\varphi f(t) = \sup_{0 < s < t} \varphi(s)f^*(s) \simeq \sup_{0 < s < t} \varphi(s)f^{**}(s) = \varphi(t)S_\varphi f^{**}(t), \quad t \in (0, R).$$

□

**Lemma 4.3.** *Let  $0 < R \leq \infty$  and let  $\varphi$  and  $\psi$  be quasiconcave functions on  $(0, R)$ . Then*

$$S_\varphi f^{**} + T_\psi f^{**} \simeq S_\varphi f + T_\psi f$$

for every measurable  $f$  if and only if both  $\varphi \in B$  and  $\psi \in B$  hold.

**Proof.** The claim is a corollary of Lemma 4.1 since

$$T_\psi f^{**} \leq T_\psi f + f^{**} \leq T_\psi f + S_\varphi f^{**}$$

and Lemma 4.2 which ensures that

$$S_\varphi f^{**} \lesssim S_\varphi f.$$

The opposite inequality and the necessity are obvious.

□

## 5. PROOF OF THE MAIN RESULTS

**Proof of Theorem 1.1.** Let us fix  $f \in \mathcal{M}(\mathcal{R}_1)$  and  $t \in (0, R)$ . We decompose  $f = f^t + f_t$  by

$$\begin{aligned} f^t(x) &= \max\{|f(x)| - f^*(t), 0\} \operatorname{sgn} f(x), \\ f_t(x) &= \min\{|f(x)|, f^*(t)\} \operatorname{sgn} f(x). \end{aligned}$$

We then have

$$(f^t)^{**}(s) \leq \frac{t}{s} f^{**}(t), \quad s \in (0, R), \tag{5.1}$$

$$(f_t)^{**}(s) \leq f^{**}(t), \quad s \in (0, R). \tag{5.2}$$

Thus

$$\begin{aligned}
(Tf)^{**}(t) &\lesssim (Tf^t + Tf_t)^{**}(t) && \text{(by quasilinearity)} \\
&\leq (Tf^t)^{**}(t) + (Tf_t)^{**}(t) && \text{(by subadditivity of }^{**}\text{)} \\
&\leq \frac{1}{\varphi(t)} \sup_{0 < s < R} \varphi(s) (Tf^t)^{**}(s) + \frac{1}{\psi(t)} \sup_{0 < s < R} \psi(s) (Tf_t)^{**}(s) \\
&= \frac{1}{\varphi(t)} \|Tf^t\|_{M_\varphi} + \frac{1}{\psi(t)} \|Tf_t\|_{M_\psi} \\
&\lesssim \frac{1}{\varphi(t)} \|f^t\|_{M_\varphi} + \frac{1}{\psi(t)} \|f_t\|_{M_\psi} \quad \text{(by the boundedness of } T \text{ on } M_\varphi \text{ and } M_\psi) \\
&= \frac{1}{\varphi(t)} \sup_{0 < s < R} \varphi(s) (f^t)^{**}(s) + \frac{1}{\psi(t)} \sup_{0 < s < R} \psi(s) (f_t)^{**}(s) \\
&= \text{I} + \text{II}.
\end{aligned}$$

Next,

$$\begin{aligned}
\text{I} &\leq \frac{1}{\varphi(t)} \sup_{0 < s < t} \varphi(s) f^{**}(s) + \frac{1}{\varphi(t)} \sup_{0 < s < t} \varphi(s) (f_t)^{**}(s) + \frac{1}{\varphi(t)} \sup_{t < s < R} \varphi(s) (f^t)^{**}(s) \\
&\leq S_\varphi f^{**}(t) + f^{**}(t) \frac{1}{\varphi(t)} \sup_{0 < s < t} \varphi(s) + f^{**}(t) \frac{t}{\varphi(t)} \sup_{t < s < R} \frac{\varphi(s)}{s} \quad \text{(by (5.2) and (5.1))} \\
&\lesssim S_\varphi f^{**}(t) + f^{**}(t) \\
&\lesssim S_\varphi f^{**}(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\text{II} &\leq \frac{1}{\psi(t)} \sup_{t < s < R} \psi(s) f^{**}(s) + \frac{1}{\psi(t)} \sup_{t < s < R} \psi(s) (f^t)^{**}(s) + \frac{1}{\psi(t)} \sup_{0 < s < t} \psi(s) (f_t)^{**}(s) \\
&\leq T_\psi f^{**}(t) + f^{**}(t) \frac{t}{\psi(t)} \sup_{t < s < R} \frac{\psi(s)}{s} + f^{**}(t) \frac{1}{\psi(t)} \sup_{0 < s < t} \psi(s) \\
&\lesssim T_\psi f^{**}(t).
\end{aligned}$$

Adding both parts together, we obtain

$$(Tf)^{**}(t) \lesssim S_\varphi f^{**}(t) + T_\psi f^{**}(t).$$

Now thanks to Lemma 4.3 we can put the double stars away and continue by

$$(Tf)^{**}(t) \lesssim S_\varphi f(t) + T_\psi f(t) \lesssim (S_\varphi f + T_\psi f)^{**}(t).$$

Now, the claim of the theorem follows by Hardy's lemma [1, Chapter 2, Corollary 4.7].  $\square$

**Remark 5.1.** Before we get to the proof of Theorem 1.3 let us first say a few words about additional assumptions (1.2) and (1.3) in the case of discontinuous quasiconcave function  $\varphi$ .

Denote  $\varphi(0+) = \lim_{t \rightarrow 0+} \varphi(t) > 0$ . Since

$$S_\varphi f(t) = \frac{1}{\varphi(t)} \sup_{0 < s < t} \varphi(s) f^*(s) \geq \frac{\varphi(0+)}{\varphi(t)} \|f\|_{L^\infty}, \quad t \in (0, R),$$

we get  $S_\varphi f(t) = \infty$  on whole  $(0, R)$  for every unbounded  $f$ . Thus, in the sake of nontriviality, we are only interested in the situation when  $\Gamma_{\phi_1}^p \hookrightarrow L^\infty$ . The embeddings of this type were studied in many papers. By methods of [2, Remark 2.3], this embedding is equivalent to  $\sup_{0 < t < R} 1/\varphi \Gamma_{\phi_1}^p < \infty$ , which rewrites as

$$\inf_{0 < t < R} \left( \int_0^t \phi_1(s) \, ds + t^p \int_t^R s^{-p} \phi_1(s) \, ds \right) > 0. \quad (5.3)$$

However, since  $\phi_1$  is assumed to be positive, (5.3) is equivalent to (1.3).

Nontriviality also depends on the interplay between quasiconcave function  $\varphi$  and weight  $\phi_2$ . Indeed,

$$\begin{aligned} \|S_\varphi f\|_{\Gamma_{\phi_2}^p}^p &= \int_0^R [(S_\varphi f)^{**}(s)]^p \phi_2(s) \, ds \\ &\simeq \int_0^R [(S_\varphi f)(s)]^p \phi_2(s) \, ds && \text{(by Lemma 4.2)} \\ &= \int_0^R \varphi^{-p}(s) \phi_2(s) \sup_{0 < y < s} [f^*(y)]^p \varphi^p(y) \, ds \\ &\geq \varphi^p(0+) \|f\|_{L^\infty}^p \int_0^R \varphi^{-p}(s) \phi_2(s) \, ds \end{aligned}$$

and as we can see the (1.2) is a necessary assumption in order to avoid the situation when  $S_\varphi f \notin \Gamma_{\phi_2}^p$  for any nontrivial  $f$ .

**Proof of Theorem 1.3.** The necessity follows by plugging the characteristic function into (1.5). As for the sufficiency let us first deal with the case of continuous  $\varphi$ . Take

an arbitrary function  $f \in \mathcal{M}(0, R)$  and estimate

$$\begin{aligned}
\|S_\varphi f\|_{\Gamma_{\phi_2}^p}^p &= \int_0^R [(S_\varphi f)^{**}(s)]^p \phi_2(s) \, ds \\
&\lesssim \int_0^R [(S_\varphi f)(s)]^p \phi_2(s) \, ds && \text{(by Lemma 4.2)} \\
&= \int_0^R \varphi^{-p}(s) \phi_2(s) \sup_{0 < y < s} [f^*(y)]^p \varphi^p(y) \, ds \\
&\simeq \int_0^R \varphi^{-p}(s) \phi_2(s) \sup_{0 < y < s} [f^*(y)]^p \int_0^y \varphi^{p-1}(t) \varphi'(t) \, dt \, ds \\
&\leq \int_0^R \varphi^{-p}(s) \phi_2(s) \sup_{0 < y < s} \int_0^y [f^*(t)]^p \varphi^{p-1}(t) \varphi'(t) \, dt \, ds \\
&= \int_0^R \varphi^{-p}(s) \phi_2(s) \int_0^s [f^*(t)]^p \varphi^{p-1}(t) \varphi'(t) \, dt \, ds \\
&= \int_0^R [f^*(t)]^p \varphi^{p-1}(t) \varphi'(t) \int_t^R \varphi^{-p}(s) \phi_2(s) \, ds \, dt.
\end{aligned}$$

Thus, we only need that

$$\int_0^R [f^*(t)]^p w(t) \, dt \lesssim \int_0^R [f^{**}(t)]^p \phi_1(t) \, dt \quad (5.4)$$

where

$$w(t) = p \varphi^{p-1}(t) \varphi'(t) \int_t^R \varphi^{-p}(s) \phi_2(s) \, ds, \quad t \in (0, R). \quad (5.5)$$

By [8, Theorem 3.2], the inequality (5.4) holds if and only if

$$\int_0^t w(s) \, ds \lesssim \int_0^t \phi_1(s) \, ds + t^p \int_t^R s^{-p} \phi_1(s) \, ds, \quad t \in (0, R), \quad (5.6)$$

which is equivalent to (1.6) by integration by parts.

For the sufficiency in the case  $\varphi$  is discontinuous, we start similarly

$$\begin{aligned}
\|S_\varphi f\|_{\Gamma_{\phi_2}^p}^p &\lesssim \int_0^R \varphi^{-p}(s) \phi_2(s) \sup_{0 < y < s} [f^*(y)]^p \left( p \int_0^y \varphi^{p-1}(t) \varphi'(t) \, dt + \varphi^p(0+) \right) \, ds \\
&\simeq \int_0^R \varphi^{-p}(s) \phi_2(s) \sup_{0 < y < s} [f^*(y)]^p \int_0^y \varphi^{p-1}(t) \varphi'(t) \, dt \, ds \\
&\quad + \|f\|_{L^\infty}^p \int_0^R \varphi^{-p}(s) \phi_2(s) \, ds.
\end{aligned}$$

The second term is estimated by a constant multiple of  $\|f\|_{\Gamma_{\phi_1}^p}^p$ , thanks to the assumptions. As for the first one, we proceed in the same way as above and again, due to [8,

Theorem 3.2], we obtain the sufficiency of (5.6) where  $w$  is defined as in (5.5). Now, by integration by parts of the left side, we get

$$\begin{aligned} \int_0^t w(s) ds &= \int_0^t \phi_2(s) ds + \varphi^p(t) \int_t^R \varphi^{-p}(s) \phi_2(s) ds \\ &\quad - \varphi^p(0+) \int_0^R \varphi^{-p}(s) \phi_2(s) ds, \quad t \in (0, R), \end{aligned}$$

and clearly (1.6) is also sufficient for (1.5) in this case.  $\square$

**Proof of Theorem 1.4.** Assume that (1.8) holds. We have

$$\begin{aligned} \|T_\psi f\|_{\Gamma_{\phi_2}^p}^p &= \int_0^R [(T_\psi f)^{**}(s)]^p \phi_2(s) ds \\ &\lesssim \int_0^R [T_\psi f(s)]^p \phi_2(s) ds + \int_0^R [f^{**}(s)]^p \phi_2(s) ds \quad (\text{by Lemma 4.1}) \\ &= \text{I} + \text{II}. \end{aligned}$$

Next,

$$\begin{aligned} \text{I} &\lesssim \int_0^R \psi^{-p}(s) \phi_2(s) \sup_{s < y < R} (\psi^p(y) - \psi^p(s)) [f^*(y)]^p ds \\ &\quad + \int_0^R \psi^{-p}(s) \phi_2(s) \sup_{s < y < R} \psi^p(s) [f^*(y)]^p ds \\ &= p \int_0^R \psi^{-p}(s) \phi_2(s) \sup_{s < y < R} [f^*(y)]^p \int_s^y \psi^{p-1}(t) \psi'(t) dt ds \\ &\quad + \int_0^R \phi_2(s) \sup_{s < y < R} [f^*(y)]^p ds \\ &\leq p \int_0^R \psi^{-p}(s) \phi_2(s) \int_s^R [f^*(t)]^p \psi^{p-1}(t) \psi'(t) dt ds \\ &\quad + \int_0^R [f^*(s)]^p \phi_2(s) ds \\ &\leq p \int_0^R [f^*(t)]^p \psi^{p-1}(t) \psi'(t) \int_0^t \psi^{-p}(s) \phi_2(s) ds dt \\ &\quad + \int_0^R [f^*(s)]^p \phi_2(s) ds \\ &= \int_0^R [f^*(t)]^p w(t) dt \end{aligned}$$

where we set

$$w(t) = \phi_2(t) + p \psi^{p-1}(t) \psi'(t) \int_0^t \psi^{-p}(s) \phi_2(s) ds, \quad t \in (0, R).$$

Now, it suffices to show that (1.8) implies

$$\int_0^R [f^{**}(t)]^p \phi_2(t) dt \lesssim \int_0^R [f^{**}(t)]^p \phi_1 dt \quad (5.7)$$

and also

$$\int_0^R [f^*(t)]^p w(t) dt \lesssim \int_0^R [f^{**}(t)]^p \phi_1 dt. \quad (5.8)$$

The embedding (5.7) holds if and only if

$$\int_0^t \phi_2(s) ds + t^p \int_t^R s^{-p} \phi_2(s) ds \lesssim \int_0^t \phi_1(s) ds + t^p \int_t^R s^{-p} \phi_1(s) ds, \quad t \in (0, R), \quad (5.9)$$

due to [4, Theorem 3.2], while (5.8) is by [8, Theorem 3.2] equivalent to

$$\int_0^t w(s) ds \lesssim \int_0^t \phi_1(s) ds + t^p \int_t^R s^{-p} \phi_1(s) ds, \quad t \in (0, R),$$

which is the same as

$$\psi^p(t) \int_0^t \psi^{-p}(s) \phi_2(s) ds \lesssim \int_0^t \phi_1(s) ds + t^p \int_t^R s^{-p} \phi_1(s) ds, \quad t \in (0, R), \quad (5.10)$$

by integration by parts. Finally, since

$$\int_0^t \phi_2(s) ds \leq \psi^p(t) \int_0^t \psi^{-p}(s) \phi_2(s) ds, \quad t \in (0, R),$$

due to the fact that  $\psi$  is increasing, (1.8) ensures both (5.9) and (5.10).

The necessity follows again by evaluating both sides of (1.7) on characteristics functions.  $\square$

**Proof of Theorem 1.2.** Let us first show that the validity of both conditions for the boundedness of  $S_\varphi$  and  $T_\psi$  on Lorentz gamma spaces (1.6) and (1.8) is equivalent to the condition (1.4). Indeed, since  $\psi(s)$  and  $s/\varphi(s)$  are both increasing we have

$$\int_0^t \phi_2(s) ds \leq \psi^p(t) \int_0^t \psi^{-p}(s) \phi_2(s) ds$$

and

$$t^p \int_t^R s^{-p} \phi_2(s) ds \leq \varphi^p(t) \int_t^R \varphi^{-p}(s) \phi_2(s) ds.$$

Our result then follows from Theorem 1.4 and Theorem 1.3 used together with Theorem 1.1.  $\square$

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